Cracking of Self-Gravitating Compact Objects with Local and Non Local Equations of State

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Abstract. We discuss the effect that small fluctuations of density and local anisotropy (principal unequal stresses) may have on the occurrence of cracking in spherical compact objects having a non local and local politropic equations of state. A non local equation of state provides, at a given point, the radial pressure not only as function of the density at that point, but its functional throughout the enclosed distribution. It is shown that departures from the equilibrium may lead to the appearance of cracking and it seams to be related not only to fluctuations on the local anisotropy but also to the density. We have found that these fluctuations should have the same sign and the effect of the perturbations in density qualitatively different to the variations in anisotropy.

1. Cracking of selfgravitating compact objects

Some years ago, Herrera and collaborators in a series of papers [1, 2, 3, 4] elaborate the concept of cracking for selfgravitating isotropic and anisotropic matter configurations. It was introduced to describe the behaviour of fluid distributions just after its departure from equilibrium, when total non-vanishing radial forces of different signs appear within the system. They state that there is a cracking whenever the radial force is directed inward in the inner part of the sphere and reverses its sign beyond some value of the radial coordinate or, when the force is directed outward in the inner part and changes sign in the outer part, we shall say that there is an overturning. This effect is related to the tidal acceleration of fluid elements [3, 5], defined by

$$a^\alpha = h^\alpha_\beta u^\gamma \left(u^\beta_{;\mu}\, h^\mu_\nu \, \delta x^\nu \right)_{;\gamma} \Leftrightarrow a^\alpha = \left[-R^\alpha_{\beta\gamma\mu} u^\beta u^\mu + h^\alpha_\beta \left(\frac{\mathrm{d}u^\beta}{\mathrm{d}s} \right)_{;\gamma} - \frac{\mathrm{d}u^\alpha}{\mathrm{d}s} \frac{\mathrm{d}u_\gamma}{\mathrm{d}s} \right] h^\gamma_\nu \, \delta x^\nu \quad (1)$$

where h^α_β denotes the projector onto the three-space orthogonal to the four-velocity u^α, δx^ν is a vector connecting the two neighbouring particles and $\frac{\mathrm{d}u^\alpha}{\mathrm{d}s} \equiv u^\mu u^\alpha_{;\mu}$. More over, defining

$$R = \frac{\mathrm{d}P_r}{\mathrm{d}r} + (\rho + P_r) \left(\frac{m + 4\pi r^3 P_r}{r\,(r - 2m)} \right) - \frac{2}{r} \left(P_\perp - P_r \right) \quad (2)$$

which is just the hydrostatic equilibrium equation that vanishes for static (or slowly evolving) configurations, and can be obtained considering spherically symmetric distribution of matter with a vanishing rotation velocity, i.e. $2\Omega^2 = \Omega^{\alpha\beta}_{\alpha\beta} = 0$. Equations (1) and (2) evaluated at the moment immediately after perturbation lead to [3, 4]

$$R = -\frac{e^{2\lambda}(\rho + P)}{e^\nu r^2} \int_0^a \mathrm{d}\tilde{r}\; e^\nu \tilde{r}^2 \frac{\mathrm{d}\Theta}{\mathrm{d}s} \quad (3)$$

where Θ represents the expansion and, $\mathrm{d}s^2 = e^{2\nu}\mathrm{d}t^2 - e^{2\lambda}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin\theta \mathrm{d}\phi^2)$, the Schwarzchild line element, has been assumed (see reference [3] for details). It can be appreciated from (3) that for the cracking to occur at some value of $0 \leq r \leq a$, it is necessary that $\frac{\mathrm{d}\Theta}{\mathrm{d}s}$ vanishes somewhere within the configuration. It also clear the non local nature of this effect and, it has been shown that even small deviations from local isotropy may lead to drastic changes in the evolution of the system as compared with the purely locally isotropic case [4].

On the other hand and concerning the nonlocal physics within general relativistic matter configurations, we have study a particular equation of state where the radial pressure $P_r(r)$ is not only a function of the energy density, $\rho(r)$, at that point but also its functional throughout the rest of the configuration. For this non local equation of state (*NLES*), any change in the radial pressure takes into account the effects of the variations of the energy density within the entire volume [6, 7, 8]. It has been shown that in the static limit the *NLES* can be written as

$$P_r(r) = \rho(r) - \frac{2}{r^3} \int_0^r \bar{r}^2 \rho(\bar{r})\; \mathrm{d}\bar{r}; \quad (4)$$

where $\mathcal{C}$ is an arbitrary integration constant. It is clear that in equation (4) a collective behavior on the physical variables $\rho(r)$ and $P_r(r)$ is present.

In the present paper we shall explore the influence of fluctuations of density and of local anisotropy have, on the stability of non-local and local politropic anisotropic matter configurations in general relativity.

This paper is organized as follows. Next Section 2 will describe non local matter configurations and state our notation. The concept of cracking for selfgravitating isotropic and anisotropic matter configurations is considered in Section 3. Finally some preliminary conclusions are displayed in Section 5

2. Compact objects with a non nocal equations of state

Following [7] we shall consider a static spherically symmetric anisotropic distribution of matter with an energy-momentum represented by $\mathbf{T}^{\mu}_{\nu} = diag\ (\rho, -P_r, -P_\perp, -P_\perp)$, where, ρ is the energy density, P_r the radial pressure and $P_\perp$ the tangential pressure. Adopting the above standard Schwarzschild coordinates (t, r, θ, ϕ) Einstein field equations can be written as

$$8\pi\rho = \frac{1}{r^2} + \frac{e^{-2\lambda}}{r}\left[2\lambda' - \frac{1}{r}\right]; \qquad -8\pi P_r = \frac{1}{r^2} - \frac{e^{-2\lambda}}{r}\left[2\nu' + \frac{1}{r}\right]$$

and

$$-8\pi P_\perp = e^{-2\lambda}\left[\frac{\lambda'}{r} - \frac{\nu'}{r} - \nu'' + \nu'\lambda' - (\nu')^2\right]$$

where primes denote differentiation with respect to r.

Now, equation (4) can re-stated as a differential equation

$$P_r(r) = \rho(r) - \frac{2}{r^3}\int_0^r \bar{r}^2\rho(\bar{r})\ \mathrm{d}\bar{r} \quad \Leftrightarrow \quad \rho - 3P_r + r\left(\rho' - P_r'\right) = 0 \tag{5}$$

and the corresponding Einstein Field Equations for anisotropic fluids with *NLES* (5) can be written as:

$$4\pi\rho = \frac{m'}{r^2}, \tag{6}$$

$$4\pi P_r = \frac{m'}{r^2} - \frac{2m}{r^3} \qquad \text{and} \tag{7}$$

$$8\pi P_\perp = \frac{m''}{r} + \frac{2\left(m'r - m\right)}{r^3}\left[\frac{m'r - m}{r - 2m} - 1\right]. \tag{8}$$

and the corresponding line element is

$$\mathrm{d}s^2 = \frac{\left(e^{2\kappa}\mathrm{d}t^2 - dr^2\right)}{\left(1 - 2\frac{m(r)}{r}\right)} - r^2\mathrm{d}\Omega^2 \tag{9}$$

where $e^{\kappa} = 1 - \frac{m(R)}{R}$ and $r = R$ the boundary of the distribution. It is clear that if the profile of the energy density, $\rho(r)$, is provided, the metric element $m(r)$ and all other physical variables (P_r and $P_\perp$) can be obtained from field equations (6), (7) and (8), respectively.

3. Cracking configurations

Now, following [4] we assume that the system having some pressure and density distributions satisfying $R = 0$, is perturbed from its hydrostatic equilibrium. Thus, fluctuations in density and pressure induce total radial forces ($R = 0$) which depending on their spatial distribution may lead to the *cracking*, i.e radial force directed inward, $R > 0$ or *overturning* directed outward, $R < 0$ of the source. Therefore, we will be looking for a change of the sign of R, beyond some value of the radial coordinate. For the cracking to occur, we will be considering, exclusively, perturbations under which the system is dynamically unstable. One way to assure this is to assume that the value of the ratio of specific heats of the fluid is not equal to the critical value required for marginal (neutral) dynamical stability. In other words, under perturbations of density and local anisotropy, we shall assume that the radial pressure of the system maintains the same radial dependence it had in equilibrium.

$$R = R(\rho_0 + \delta\rho, P_r, \Delta_0 + \delta\Delta) \equiv \underbrace{R_0(\rho_0, P_r, \Delta_0)}_{=0} + \tilde{R}(\rho_0, P_r, \Delta_0, \delta\rho, \delta\Delta) \qquad (10)$$

The inability to adapt its radial pressure to the perturbed situation is equivalent to assuming that the pressure-density relation (the ratio of specific heats) never reaches the value required for neutral equilibrium

4. The modeling performed

We shall compare perturbed models starting with the same density profile for *local* and *non-local*. For non local models we use three of the solutions presented in [7] and for *local* ones we shall consider politropic equations of state for anisotropic matter configurations, i.e.

$$\rho(r) \rightarrow \left\{ \begin{array}{rcl} \rho(r) - \frac{2}{r^3}\int_0^r \bar{r}^2\rho(\bar{r})\,\mathrm{d}\bar{r} &=& P_r(r) \\ \\ K\rho^{\Gamma}(r) = && P_r(r) \end{array} \right. \quad \text{for } R = 0 \Rightarrow \left\{ \begin{array}{l} \Delta_{NL} \\ \\ \Delta_{Politropic} \end{array} \right. \qquad (11)$$

and in each case, for the configurations in equilibrium, the local anisotropy, $\Delta \propto \frac{P_\perp - P_r}{r}$, is obtained.

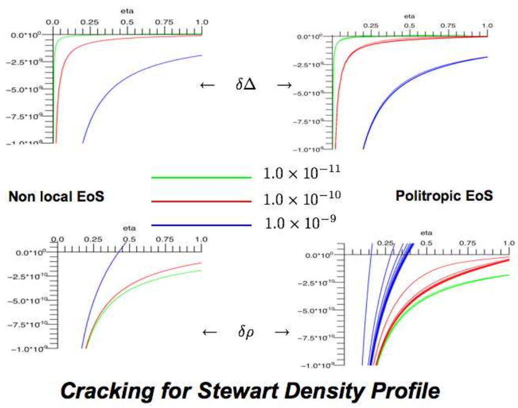

Figure 1. The induced total radial force for Steward's density profile. Left plates correspond to non local models, while right ones represent a family of politropic equations of state for different values of the politropic index, n.

In order to illustrate the above procedure we shall work out several examples from [7]. The first density profile comes from a density profile proposed by B. W. Stewart [9], to describe anisotropic conformally flat static bounded configurations:

$$\rho_S = \frac{1}{8\,\pi r^2}\frac{\left(e^{2Kr}-1\right)\left(e^{4Kr}+8Kre^{2Kr}-1\right)}{\left(e^{2Kr}+1\right)^3} \quad \Leftrightarrow \quad m_S = \frac{r}{2}\left(\frac{e^{2Kr}-1}{e^{2Kr}+1}\right)^2 \ , \qquad (12)$$

Density Profile	μ	$M\ (M_{\odot})$	z_a	$\rho_a \times 10^{14}\,(gr/cm^3)$	$\rho_c\ \times 10^{15}\ (gr/cm^3)$
Stewart	0.32	2.15	0.6	6.80	1.91
Gokhroo-Mehra	0.40	2.80	1.2	8.84	1.99
Wyman	0.38	2.54	1.0	8.04	3.04

Table 1. All parameters have been chosen to represent a possible compact object with $a = 10$ Km. and the corresponding mass function satisfying the physical and energy conditions

with $K = const.$

The density profile of the second example is due, originally, to P.S. Florides [10], but also by Stewart [9] and more recently by M. K. Gokhroo and A. L. Mehra [11]. This solution, represents densities and pressures which, under particular circumstances [12], give rise to an equation of state similar to the Bethe-Börner-Sato newtonian equation of state for nuclear matter [13].

$$\rho_{GM} = \frac{\sigma}{8\pi}\left[1 - K\frac{r^2}{a^2}\right] \qquad \Leftrightarrow \qquad m_{GM} = \frac{\sigma\, r^3}{6}\left[1\ -\frac{3\,K}{5}\frac{r^2}{a^2}\right]\ , \qquad (13)$$

with σ and $K = const.$

The last density profile corresponds to a solution originally proposed M. Wyman [14] and is written as

$$\rho_W = -\frac{C}{8\,\pi}\frac{K\,(3+5\,x)}{(1+3\,x)^{\frac{5}{3}}} \qquad \Leftrightarrow \qquad m_W = -\frac{1}{2C^{\frac{1}{2}}}\frac{K x^{\frac{3}{2}}}{(1+3x)^{\frac{2}{3}}}\ , \qquad (14)$$

with $x = C\,r^2$ and $K, C = const.$

In table 1 we have summarized the physical parameters involved in the above models. They are: the mass, M, in terms of solar mass $M_{\odot}$, the gravitational potential at the surface μ, the boundary redshift z_a, the surface density ρ_a and the central density ρ_c.

4.1. Cracking Non local models

For all de above density profiles, we calculate the induced total radial force from (10). Thus, for the Steward model, (12), can be written as

$$\widetilde{\mathcal{R}}_{S-NL} = \frac{\mu\,(e^{\mu\eta}-1)}{(e^{\mu\eta}+1)}\delta\rho - \frac{2}{\eta}\delta\Delta \qquad (15)$$

where, from now on, we have denoted $\eta = \frac{r}{a}$ and $\mu = \frac{M}{a}$ and a corresponds to the boundary of the distribution. Figure 1 displays it for several of values of density perturbations.

Again, for the second example (13) the perturbed radial force (10) is

$$\widetilde{\mathcal{R}}_{GM-NL} = \frac{\mu\eta\,(10-12\eta^2 K)}{(5-3K-10\mu\eta^2 K)}\delta\rho - \frac{2}{\eta}\delta\Delta \qquad (16)$$

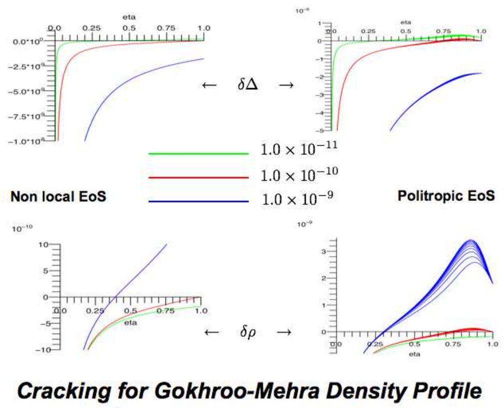

Figure 2. The induced total radial force for Gokhroo-Mehra's density profile. Left plates correspond to non local models, while right ones represent a family of politropic equations of state for different values of the politropic index, n.

and it is shown in Figure 2

Finally from (14), the expresion for $\widetilde{\mathcal{R}}_{W-NL}$ can be written as

$$\widetilde{\mathcal{R}}_{W-NL} = -\frac{\eta K(\eta^2+1)}{\left(K\eta^2 + (1+3\eta^2)^{2/3}\right)(1+3\eta^2)}\delta\rho - \frac{2}{\eta}\delta\Delta \tag{17}$$

and it is plotted in Figure 3.

4.2. Cracking politropic models

As we have pointed out in (11) we are going to compare the cracking for local and non local models, and we are going to model local equations of state by using a politropic relation between pressure and density, [15, 16, 17], i.e.

$$P_r(r) = \kappa\rho(r)^{1+\frac{1}{n}}, \quad \text{if } \rho(r) = \rho_0\theta(r) \quad \Rightarrow \rho(r) = \kappa\rho_0^{\frac{1}{n}}\rho_0\theta(r)^{1+\frac{1}{n}}, \tag{18}$$

The expresion for $\widetilde{\mathcal{R}}_{S-Pol}$ for Stewart model, (12), can be written as

$$\widetilde{\mathcal{R}}_{S-Pol} = \frac{\left((\sigma\Xi\left(e^{2c\eta}+4c\eta e^{c\eta}-1\right)+1)\left(1-e^{-c\eta}\right)+e^{c\eta}\left(e^{c\eta}-1\right)\right)}{8\eta\left(e^{c\eta}+1\right)}\delta\rho - \frac{2\delta\Delta}{\eta} \tag{19}$$

where

$$\Xi = \frac{\left(e^{c\eta}-1\right)\left(e^{2c\eta}+4c\eta e^{c\eta}-1\right)^{\frac{1}{n}}}{\eta^2\left(e^{c\eta}+1\right)^3} \qquad \text{and} \qquad c = \ln\left[\frac{1+(\frac{2M}{a})^{\frac{1}{2}}}{1-(\frac{2M}{a})^{\frac{1}{2}}}\right] \tag{20}$$

The total induced radial force for the politropic Gokhroo-Mehra model is

$$\widetilde{\mathcal{R}}_{GM-Pol} = -\frac{\mu\eta\left(3\eta^2-5-15\sigma\left(1-\eta^2\right)^{\frac{2n+1}{n}}\right)}{2(1-5\mu\eta^2+3\mu\eta^4)}\delta\rho - \frac{2\delta\Delta}{\eta} \tag{21}$$

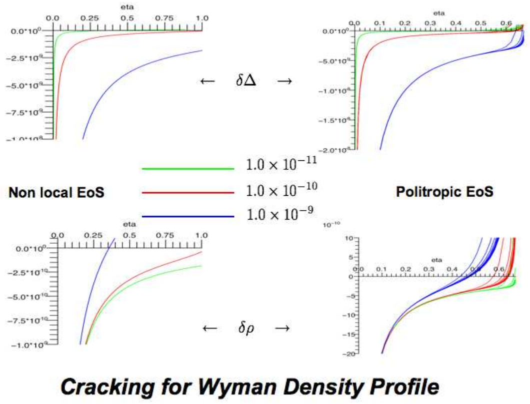

Figure 3. The induced total radial force for Wyman's density profile. Left plates correspond to non local models, while right ones represent a family of politropic equations of state for different values of the politropic index, n.

Finally, $\widetilde{\mathcal{R}}_{W-Pol}$ can be arranged as

$$\widetilde{\mathcal{R}}_{W-Pol} = \frac{3k\eta\left(\sigma 125^{\left(1+\frac{1}{n}\right)}\frac{(\eta^2-1)}{(9\eta^2-5)}\left(\frac{\eta^2-1}{(9\eta^2-5)(125-225\eta^2)^{2/3}}\right)^{\frac{1}{n}}-25\right)}{10\left((125-225\eta^2)^{2/3}-15k\eta^2\right)}\delta\rho - \frac{2\delta\Delta}{\eta} \qquad (22)$$

5. Results and conclusions

We have explored the influence of density and anisotropy fluctuations on the stability of some matter configurations in General Relativity. It have been found that departures from the equilibrium may lead to the appearance of cracking (or overturning) and it is related not only to fluctuations on the local anisotropy but also to the density. In fact, at least for the families of equation of state we consider, in order to have a cracking within the configuration, both perturbations should have the same sign, i.e. $\delta\rho > 0 \wedge \delta\Delta > 0$ or $\delta\rho < 0 \wedge \delta\Delta < 0$.

To study the effect of these perturbations we implement the concept of cracking for selfgravitating anisotropic matter configurations, developed by Herrera and collaborators in a series of papers [1, 2, 3, 4]. It was introduced to describe the behaviour of fluid distributions just after its departure from equilibrium, when total non-vanishing radial forces of different signs appear at some point within the system. As we have stated in Section 3, whenever the perturbed radial force (10) is directed inward in the inner part of the sphere and reverses its sign at some point, we will have a *cracking* or, when the force is directed outward in the inner part and changes sign in the outer part, we shall say that there is an *overturning*.

In order to compare the effects of these type of perturbations, we have considered two families of equations of state which differ in its relation between radial pressure and density. The first type, known as non local equations of state (see references [6, 7, 8]), provides a relation between radial pressure and density not only as its function at a particular point, but a functional throughout the enclosed distribution. The second family of equation of state is the politropic anisotropic equation (18). The two types of models have the same physical description displayed in Table 1 and equivalent densities profiles, i.e. (12), (13) and (14).

As it can appreciated from figures 1, 2 and 3 the effect of density and local anisotropy perturbations are qualitatively different. It is evident that the greater the fluctuation in the density we have, the larger the possibility for the sphere to crack. Inversely, the stability of the configurations reveals to be very sensitive to smaller perturbations in the anisotropy of the stresses. Very small perturbations in the anisotropy are more effective to generate instability in the configurations. More over, from the expressions of the total induced radial force (equations (15), (16), (17), (19), (21) and (22)), it is clear that the change of its sign emerges from the density perturbation (and not form the fluctuations in anisotropy) because the contribution of the anisotropy to the instability is equivalent for all the models. Nevertheless, there should be some perturbation in the anisotropy for a cracking to occur.

These are preliminary results that should be further explored and forthcoming results will be reported elsewhere. It is clear that our comprehension of the behavior of highly compact stars is intimately related to the understanding of the physics at supranuclear densities which is, today, essentially unknown. Thus, having this uncertainty in mind, we can explore what is allowed by the present laws of physics, for these particular equations of state, within the framework of the theory of General Relativity.

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