Revista Notas de Matemática
Vol.7(1), No. 304, 2011, pp. 57-65
http://www.saber.ula.ve/notasdematematica
Pre-prints
Departamento de Matemáticas
Facultad de Ciencias
Universidad de Los Andes

Special motions for spacelike curve in Minkowski 3-space

Naser Masrouri and Yusuf Yayli

Abstract

Existence of acceleration pole points in special Frenet and Bishop motions for spacelike curve with a spacelike binormal in Minkowski 3-space E_1^3 are dependence into that, the curve α is not a general helix or planar. The ratio of torsion and curvature is by taking as a constant or non constant in our study. Then we show that, if the ratio of curvatures is constant, then there is not acceleration pole points of motion.

AMS subject classifications. Primary 53A04; 53A17

1 Preliminaries

Let R^3 be the real vector space with its usual vector structure. The Minkowski 3-space is the metric space $E_1^3 = (R^3, \langle,\rangle_L)$, where the metric $\langle,\rangle_L$ is given by

$$\langle x,y\rangle_L = x_1y_1 + x_2y_2 - x_3y_3 : \ x = (x_1,x_2,x_3), \ y = (y_1,y_2,y_3)$$

The metric $\langle,\rangle_L$ is called the Lorentzian metric [4, 6].
A vector $x \in E_1^3$ is called:
i) Spacelike if $\langle x,x\rangle_L > 0$ or $x = 0$,
ii) Timelike if $\langle x,x\rangle_L < 0$,
iii) Null (lightlike) if $\langle x,x\rangle_L = 0$ and $x \neq 0$.
Denote by $\{T, N, B\}$ the moving Frenet frame and the moving Bishop frame along the regular curve $\alpha = \alpha(t)$ that are parameterized by the length- arc parameter t The Frenet trihedron consists of the tangent vector T, the principle normal vector N and the binormal vector B, and the Bishop trihedron consists of the tangent vector T, the 1^{st} principle normal vector N_1 and 2^{nd} principle normal vector N_2, which are three mutually orthogonal axes.

If α is a spacelike curve with a spacelike binormal, then this set of orthogonal unit vectors, known as the Frenet- serret frame has the following properties:

$$\dot{T} = \kappa N,\ \dot{N} = \kappa T + \tau B,\ \dot{B} = \tau N$$
$$\langle T, T\rangle_L = 1,\ \langle N, N\rangle_L = -1,\ \langle B, B\rangle_L = 1$$

[1,5,6]. In this formulas, the normal vector is $N = \frac{\dot{T}}{\kappa}$, where $\kappa = \sqrt{-\langle \dot{T}, \dot{T}\rangle_L}$ is the curvature of α. The binormal vector is $B = T \wedge_L N$, which is a spacelike vector and the torsion of α is $\tau = \langle \dot{N}, B\rangle$.

The Bishop frame is an alternative approach to defining a moving frame that is well defined even when the spacelike curve with a spacelike binormal has vanishing second derivative. We can parallel transport an orthonormal frame along a spacelike curve with a spacelike binormal simply by parallel transporting each component of the frame. The parallel transport frame is based on the observation that, while $T(t)$ for a given spacelike curve with a spacelike binormal model is unique, we may choose any convenient arbitrary basis $(N_1(t), N_2(t))$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $T(t)$ at each point. If the derivatives of $(N_1(t), N_2(t))$ depend only on $T(t)$ and not each other we can make $N_1(t)$ and $N_2(t)$ vary smoothly throughout the path regardless of the curvature. Therefore, we have the alternative frame equations:

$$\dot{T} = \kappa_1 N_1 - \kappa_2 N_2,\ \dot{N}_1 = \kappa_1 T, \dot{N}_2 = \kappa_2 T$$
$$\langle T, T\rangle_L = 1,\ \langle N, N\rangle_L = -1,\quad \langle N_2, N_2\rangle_L = 1$$
$$\kappa(t) = \sqrt{\left|\kappa_1^2 - \kappa_2^2\right|}, \tau(t) = \frac{d\theta(t)}{dt},\quad \theta(t) = \arctan h\left(\frac{\kappa_2}{\kappa_1}\right)$$

[2,3]. So that κ_1 and κ_2 effectively correspond to a Cartesian coordinate system for the polar coordinates κ, θ with $\theta = \int \tau(t)dt$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0, which disappears from τ (and hence from the Frenet frame) due to the differentiation.

2 Introduction

In one parameter motion of a body in Lorentz-Minkowski 3-space is generated by the transformation

$$f : E_1^3 \longrightarrow E_1^3$$

$$X \longrightarrow f(X) = Y = AX + C \tag{1}$$

Where $A \in SO_1(3)$ and X, Y, C are 3×1 real matrices and

$$SO_1(3) = \left\{ A \in R_3^3 \middle| \det A = 1, \quad A^t \varepsilon A = \varepsilon, \quad \varepsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

A, C are C^∞ functions of a real parameter t, X and Y corresponding to the position vectors of the same point X, with respect to the orthonormal coordinate systems of the moving space H and the fixed space H_0, respectively. At the initial time $t = t_0$ we consider the coordinate system of H_0 and H are coincident [2,6].

In the special Frenet and Bishop motions, A matrix is $[T\ N\ B]$ and $[T\ N_1\ N_2]$ respectively.

In this paper, we first find a geometrical meaning for $\det \dot{A}$, $\det \ddot{A}$ and $\det \dddot{A}$. The 1$^{\text{st}}$ order velocity of a fixed point X is $\dot{Y} = \dot{A}X + \dot{C}$ and for the 2$^{\text{nd}}$ and 3$^{\text{rd}}$ order velocity of this point, give us $\ddot{Y} = \ddot{A}X + \ddot{C}$ and $\dddot{Y} = \dddot{A}X + \dddot{C}$ respectively.

$\dot{Y}$ is the sliding velocity and $\ddot{Y}$ and $\dddot{Y}$ are the 1$^{\text{st}}$ and 2$^{\text{nd}}$ sliding acceleration of the point X respectively. We will show that existence of the 1$^{\text{st}}$ and 2$^{\text{nd}}$ acceleration pole points by the solution of the $\ddot{A}X + \ddot{C} = 0$ and $\dddot{A}X + \dddot{C} = 0$ systems. The solution of these systems depend on $\det \ddot{A}$ and $\det \dddot{A}$.

3 Acceleration Pole Points In Frenet Motion

Definition 3.1 *The first derivation of (1), with respect to t, we have*

$$\dot{Y} = \dot{A}X + \dot{C} + A\dot{X}$$

Where $\dot{Y}$ is the absolute velocity, $\dot{A}X + \dot{C}$ is the sliding velocity and $A\dot{X}$ is the relative velocity of the point X. The solution vector X of the system $\dot{A}X + \dot{C} = 0$ is the position vector of the point which may be considered as a fixed point of H_0 and H at the same time t. These points are called instantaneous pole points at the time t. The sliding velocity of a fixed point X in moving space H is

$$\dot{Y} = \dot{A}X + \dot{C} \tag{2}$$

and for the 2nd order velocity (or the 1st order sliding acceleration) of this point, (2) gives us

$$\ddot{Y} = \ddot{A}X + \ddot{C} \tag{3}$$

and for the 3^{rd} order velocity (or the 2^{nd} sliding acceleration) of this point, (3) gives us

$$\dddot{Y} = \dddot{A}\,X + \dddot{C} \tag{4}$$

By using the Frenet formulas and

$$A = \begin{bmatrix} T & N & B \end{bmatrix},\ \ \dot{A} = \begin{bmatrix} \dot{T} & \dot{N} & \dot{B} \end{bmatrix},\ \ \ddot{A} = \begin{bmatrix} \ddot{T} & \ddot{N} & \ddot{B} \end{bmatrix},\ \ \dddot{A} = \begin{bmatrix} \dddot{T} & \dddot{N} & \dddot{B} \end{bmatrix}$$

we can give,

$$\begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

$$\det \dot{A} = \begin{vmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{vmatrix}. \det A = 0.$$

Then the system $\dot{A}X + \dot{C} = 0$ has not unique solution. So, the Frenet motion has not pole point.

3.1 1^{st} acceleration pole points in Frenet motion

The discussion of existence of the 1^{st} acceleration poles and the 1^{st} acceleration axodes is the discussion of the solution of the system

$$\ddot{A}X + \ddot{C} = 0 \tag{5}$$

The solution of the system of (5) depends on $\det \ddot{A}$.

Theorem 3.2 *The Spacelike curve with a spacelike binormal $\alpha(t)$ is not general helix; iff the Frenet motion has a 1^{st} acceleration pole point in the moving space H; $X = -(\ddot{A})^{-1}\ddot{C}$.*

Proof. If $\{T, N, B\}$ is an adapted Frenet frame, then we have

$$\ddot{T} = \kappa^2 T + \dot{\kappa} N + \kappa\tau B$$
$$\ddot{N} = \dot{\kappa} T + (\kappa^2 + \tau^2) N + \dot{\tau} N$$
$$\ddot{B} = \kappa\tau T + \dot{\tau} N + \tau^2 B$$

So, we obtain

$$\begin{bmatrix} \ddot{T} \\ \ddot{N} \\ \ddot{B} \end{bmatrix} = \begin{bmatrix} \kappa^2 & \dot{\kappa} & \kappa\tau \\ \dot{\kappa} & (\kappa^2 + \tau^2) & \dot{\tau} \\ \kappa\tau & \dot{\tau} & \tau^2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

By using $A = [T\ N\ B] \in SO_1(3)$, we get

$$\det \ddot{A} = \begin{vmatrix} \kappa^2 & \dot{\kappa} & \kappa\tau \\ \dot{\kappa} & (\kappa^2+\tau^2) & \dot{\tau} \\ \kappa\tau & \dot{\tau} & \tau^2 \end{vmatrix} = -\left[\kappa^2 \left(\frac{\tau}{\kappa}\right)^{\cdot}\right]^2 \tag{6}$$

Obviously as a consequence of equation (6) we have the following:

$$\det \ddot{A} = 0 \Leftrightarrow \frac{\tau}{\kappa} = \text{ constant}$$

From this case we obtain that at any moment t, if the curve $\alpha(t)$ is a generalized helix then the solution systems of (5) are not unique in fixed space H_0. The Frenet motion $Y = AX + C$ has not the 1^{st} acceleration pole point. If $\det \ddot{A} \neq 0$ then $\alpha(t)$ is not general helix and Frenet motion has a 1^{st} acceleration pole point, $X = -(\ddot{A})^{-1}\ddot{C}$. ■

3.2 2^{nd} acceleration pole points in Frenet motion

The discussion of existence of the 2^{nd} acceleration pole points and the 2^{nd} acceleration axodes is the discussion of the solution of the system

$$\dddot{A}\, X + \dddot{C} = 0 \tag{7}$$

Theorem 3.3 *If the spacelike curve with a spacelike binormal curve $\alpha(t)$ is a generalized helix; then the Frenet motion has not a 2^{nd} acceleration pole point in fixed space H_0.*

Proof. If T, N and B is an adapted Frenet frame, then we have;

$$\begin{aligned} \dddot{T} &= (3\kappa\dot{\kappa})T + (\kappa^3 + \kappa\tau^2 + \ddot{\kappa})N + (\kappa\dot{\tau} + 2\dot{\kappa}\tau)B \\ \dddot{N} &= (\kappa^3 + \kappa\tau^2 + \ddot{\kappa})T + 3(\kappa\dot{\kappa} + \tau\dot{\tau})N + (\tau^3 + \kappa^2\tau + \ddot{\tau})B \\ \dddot{B} &= (2\kappa\dot{\tau} + \dot{\kappa}\tau)T + (\tau^3 + \kappa^2\tau + \ddot{\tau})N + (3\tau\dot{\tau})B \end{aligned}$$

So, we obtain

$$\begin{bmatrix} \dddot{T} \\ \dddot{N} \\ \dddot{B} \end{bmatrix} = \begin{bmatrix} (3\kappa\dot{\kappa}) & (\kappa^3 + \kappa\tau^2 + \ddot{\kappa}) & (\kappa\dot{\tau} + 2\dot{\kappa}\tau) \\ (\kappa^3 + \kappa\tau^2 + \ddot{\kappa}) & 3(\kappa\dot{\kappa} + \tau\dot{\tau}) & (\tau^3 + \kappa^2\tau + \ddot{\tau}) \\ (2\kappa\dot{\tau} + \dot{\kappa}\tau) & (\tau^3 + \kappa^2\tau + \ddot{\tau}) & (3\tau\dot{\tau}) \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

By using $\det A = 1$, we get

$$\det \dddot{A} \quad = \quad 3\kappa^2 \left(\frac{\tau}{\kappa}\right)^{\cdot} \left[-2\kappa^2 \left(\frac{\tau}{\kappa}\right)^{\cdot} (\kappa\dot{\kappa} + \tau\dot{\tau})(\kappa\dot{\tau} - \dot{\kappa}\tau) + (\kappa^2 + \tau^2)(\kappa\ddot{\tau} - \ddot{\kappa}\tau)\right]$$

$$- \quad 3\dot{\kappa}^2 \left(\frac{\dot{\tau}}{\dot{\kappa}}\right)^{\cdot} (\kappa\ddot{\tau} - \ddot{\kappa}\tau) \tag{8}$$

As a consequence of equation of (7) we have the following:

Because $\alpha(t)$ is a general helix, then we can write

$$\left(\frac{\tau}{\kappa}\right)^{\cdot} = 0 \text{ and } \left(\frac{\dot{\tau}}{\dot{\kappa}}\right)^{\cdot} = 0$$

Thus $\det \dddot{A}$.

From this case we obtain, at any time t, the curve $\alpha(t)$ is a generalized helix, and the solution of system (7) are not unique and in fixed space H_0, the Frenet motion $Y = AX + C$ has not the 2^{nd} acceleration pole point. ■

4 Acceleration Pole Points In Bishop Motion

By using the Bishop formulas and

$$A = \begin{bmatrix} T & N_1 & N_2 \end{bmatrix}, \quad \dot{A} = \begin{bmatrix} \dot{T} & \dot{N}_1 & \dot{N}_2 \end{bmatrix},$$
$$\ddot{A} = \begin{bmatrix} \ddot{T} & \ddot{N}_1 & \ddot{N}_2 \end{bmatrix}, \quad \dddot{A} = \begin{bmatrix} \dddot{T} & \dddot{N}_1 & \dddot{N}_2 \end{bmatrix}$$

we can give,

$$\begin{bmatrix} \dot{T} \\ \dot{N}_1 \\ \dot{N}_2 \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & -\kappa_2 \\ \kappa_1 & 0 & 0 \\ \kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix},$$
$$\det \dot{A} = \begin{vmatrix} 0 & \kappa_1 & -\kappa_2 \\ \kappa_1 & 0 & 0 \\ \kappa_2 & 0 & 0 \end{vmatrix} = 0.$$

Then the system $\dot{A}X + \dot{C} = 0$ has not unique solution. So, the Bishop motion has not pole point.

4.1 1^{st} acceleration pole points in Bishop motion

Theorem 4.1 *The spacelike curve with a spacelike binormal $\alpha(t)$ is not planar in the moving space H iff The Bishop motion has a 1^{st} acceleration pole point; $X = -(\ddot{A})^{-1}\ddot{C}$,*

Proof. If $\{T, N_1, N_2\}$ is an adapted Bishop frame, then we have

$$\ddot{T} = (\kappa_1^2 - \kappa_2^2)T + \dot{\kappa}_1 N_1 - \dot{\kappa}_2 N_2$$
$$\ddot{N}_1 = \dot{\kappa}_1 T + \kappa_1^2 N_1 - \kappa_1\kappa_2 N_2$$
$$\ddot{N}_2 = \dot{\kappa}_2 T + \kappa_1\kappa_2 N_1 - \kappa_2^2 N_2$$

So, we obtain

$$\begin{bmatrix} \ddot{T} \\ \ddot{N}_1 \\ \ddot{N}_2 \end{bmatrix} = \begin{bmatrix} (\kappa_1^2 - \kappa_2^2) & \dot{\kappa}_1 & -\dot{\kappa}_2 \\ \dot{\kappa}_1 & \kappa_1^2 & -\kappa_1\kappa_2 \\ \dot{\kappa}_2 & \kappa_1\kappa_2 & -\kappa_2^2 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}$$

By using $A = \begin{bmatrix} T & N_1 & N_2 \end{bmatrix}$, we get

$$\det \ddot{A} = \begin{vmatrix} (\kappa_1^2 - \kappa_2^2) & \dot{\kappa}_1 & -\dot{\kappa}_2 \\ \dot{\kappa}_1 & \kappa_1^2 & -\kappa_1\kappa_2 \\ \dot{\kappa}_2 & \kappa_1\kappa_2 & -\kappa_2^2 \end{vmatrix} = \left[\kappa_1^2 \left(\frac{\kappa_2}{\kappa_1}\right)^{\cdot}\right]^2 \tag{9}$$

By using equations:

$$\kappa(t) = \sqrt{|\kappa_1^2 - \kappa_2^2|}, \quad \tau(t) = \frac{d\theta(t)}{dt}, \quad \theta(t) = \arctan h\left(\frac{\kappa_2}{\kappa_1}\right)$$

we have,

$$\begin{aligned} \kappa^2 &= \left|\kappa_1^2 - \kappa_2^2\right| \\ \frac{\kappa_2}{\kappa_1} &= \tan h\theta \quad \Rightarrow \quad \left(\frac{\kappa_2}{\kappa_1}\right)^{\cdot} = (1 - \tan^2 h\theta)\frac{d\theta}{dt} \\ &\qquad\quad \Rightarrow \quad \left(\frac{\kappa_2}{\kappa_1}\right)^{\cdot} = \left(1 - \frac{\kappa_2^2}{\kappa_1^2}\right)\tau = \left(\frac{\kappa_1^2 - \kappa_2^2}{\kappa_1^2}\right)\tau \\ &\qquad\quad \Rightarrow \quad \left(\frac{\kappa_2}{\kappa_1}\right)^{\cdot} = \pm\left(\frac{\kappa^2}{\kappa_1^2}\right)\tau \\ &\qquad\quad \Rightarrow \quad \kappa_1^2\left(\frac{\kappa_2}{\kappa_1}\right)^{\cdot} = \pm\kappa^2\tau \end{aligned} \tag{10}$$

Obviously as a consequence of equations (9) and (10) we have the following:

$$\det \ddot{A} = \kappa^4\tau^2 \tag{11}$$

As a consequence of equation of (11) we have the following:

$$\det \ddot{A} = 0 \Leftrightarrow \tau = 0$$

From this case we obtain, the solution systems of (5) are not unique in fixed space H_0 if and only if, at any time t, the curve $\alpha(t)$ is a plane. So that, the Bishop motion $Y = AX + C$ has not the 1^{st} acceleration pole point.

If $\det \ddot{A} \neq 0$ then $\alpha(t)$ is not plane. ■

4.2 2^{nd} acceleration pole points in Bishop motion

Theorem 4.2 *The spacelike curve with a spacelike binormal $\alpha(t)$ is a plane $\Rightarrow$ in fixed space H_0, the Bishop motion has not a 2^{nd} acceleration pole point.*

Proof. If T, N_1 and N_2 is an adapted Bishop frame, then we have;

$$\begin{aligned}
\dddot{T} &= 3(\kappa_1\dot{\kappa}_1 - \kappa_2\dot{\kappa}_2)T + (\kappa_1^3 - \kappa_1\kappa_2^2 + \ddot{\kappa}_1)N_1 + (\kappa_2^3 - \kappa_1^2\kappa_2 - \ddot{\kappa}_2)N_2 \\
\dddot{N}_1 &= (\kappa_1^3 - \kappa_1\kappa_2^2 + \ddot{\kappa}_1)T + (3\kappa_1\dot{\kappa}_1)N_1 - (\kappa_1\dot{\kappa}_2 + 2\dot{\kappa}_1\kappa_2)N_2 \\
\dddot{N}_2 &= (-\kappa_2^3 + \kappa_1^2\kappa_2 + \ddot{\kappa}_2)T + (\kappa_2\dot{\kappa}_1 + 2\dot{\kappa}_2\kappa_1)N_1 - (3\kappa_2\dot{\kappa}_2)N_2
\end{aligned}$$

$$\begin{bmatrix} \dddot{T} \\ \dddot{N}_1 \\ \dddot{N}_2 \end{bmatrix} = \begin{bmatrix} 3(\kappa_1\dot{\kappa}_1 - \kappa_2\dot{\kappa}_2) & (\kappa_1^3 - \kappa_1\kappa_2^2 + \ddot{\kappa}_1) & (\kappa_2^3 - \kappa_1^2\kappa_2 - \ddot{\kappa}_2) \\ (\kappa_1^3 - \kappa_1\kappa_2^2 + \ddot{\kappa}_1) & (3\kappa_1\dot{\kappa}_1) & -(\kappa_1\dot{\kappa}_2 + 2\dot{\kappa}_1\kappa_2) \\ (-\kappa_2^3 + \kappa_1^2\kappa_2 + \ddot{\kappa}_2) & (\kappa_2\dot{\kappa}_1 + 2\dot{\kappa}_2\kappa_1) & -(3\kappa_2\dot{\kappa}_2) \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}$$

$$\begin{aligned}
\det \dddot{A} &= \det(\dddot{T}, \dddot{N}, \dddot{B}) \\
&= 3(2\kappa\dot{\kappa}\tau + \kappa^2\dot{\tau})\left(-\kappa^4\tau + \dot{\kappa}_1^2\left(\frac{\dot{\kappa}_2}{\dot{\kappa}_1}\right)^{\cdot}\right) + 6\kappa^5\dot{\kappa}\tau^2 \qquad (12)
\end{aligned}$$

As a consequence of equation of (12) we have the following:

$$\tau = 0 \;\Rightarrow\; \det \dddot{A} = 0$$

From this case we obtain, if at any time t, the curve $\alpha(t)$ is a plane, then the solution of system (7) are not unique in fixed space H_0 and the Bishop motion $Y = AX + C$ has not the 2^{nd} acceleration pole point. ■

References

[1] A.Gray: *Modern Differential Geometry.*Cre pres, 1998.

[2] B.Bukcu, M. K. Karacabn: *Bioshop Frame of the Spacelike curve with a Spacelike Binormal in Minkowski 3-space.* (submitted for pubication).

[3] Bishop, L.R: *There is More Than One Way to Frame a Curve.* Amer.Math .Monthly (1975), Vol.82, Issue 3, 246-251.

[4] B.O'Neill: *Semi Riemannian Geometry with Application to General Relativity*. Academic Press, New York, 1983.

[5] T.Weinstein: *An Introduction to Lorentz Surfaces*. Walter de Gruyter, 1996.

[6] R.Lopez: *Differential Geometry of Curves and Surfaces in Lorentz-Minkowski space*. Departamento de Geometria y Topologia Universidad de Granada 18071 Granada, Espana, 2008.

Naser Masrouri
Ankara University, Faculty of Science,
Department of Mathematics
Tandoğan, Ankara, Turkey
e-mail: n.masrouri@gmail.com

Yusuf Yayli
Ankara University, Faculty of Science,
Department of Mathematics
Tandoğan, Ankara, Turkey
e-mail: yayli@science.ankara.edu.tr