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ON CHARACTERIZATION BERTRAND MATE OF TIMELIKE BIHARMONIC CURVES IN THE LORENTZIAN Heis^3

Talat Körpınar, Essin Turhan, Iqbal H. Jebril

Abstract

In this paper, we study non-geodesic timelike biharmonic curves and we construct parametric equations for Bertrand mate of timelike biharmonic curves in the Lorentzian Heisenberg group Heis

key words. Heisenberg group, Bertrand curve, biharmonic curve, helices.

1 Introduction

Bertrand curves discovered by J. Bertrand in 1850 are one of the important and interesting topic of classical special curve theory. A Bertrand curve is defined as a special curve which shares its principal normals with another special curve (called Bertrand mate).

On the other hand, Eells and Sampson also envisaged some generalizations and defined biharmonic maps $\varphi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds as critical points of the bienergy functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v_g,$$

where $\tau(\varphi) = \mathrm{trace}\nabla d\varphi$ is the tension field of $\mathcal{J}$ that vanishes on harmonic maps [6]. The Euler-Lagrange equation corresponding to E_2 is given by the vanishing of the bitension field

$$\tau_2(\varphi) = -\mathcal{J}^{\varphi}(\tau(\varphi)) = -\Delta\tau(\varphi) - \mathrm{trace}R^N(d\varphi, \tau(\varphi))\, d\varphi, \tag{1.1}$$

where $\mathcal{J}^{\varphi}$ is the Jacobi operator of φ . The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^{\varphi}$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In this paper, we study non-geodesic timelike biharmonic curves and we construct parametric equations for Bertrand mate of timelike biharmonic curves in the Lorentzian Heisenberg group Heis^3.

2 The Lorentzian Heisenberg Group Heis^3

The Lorentzian Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multiplication:

$$(\overline{x},\overline{y},\overline{z})(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\overline{x}y+x\overline{y}).$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Lorentz metric g is given by

$$g=-dx^2+dy^2+(xdy+dz)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1=\frac{\partial}{\partial z},\quad \mathbf{e}_2=\frac{\partial}{\partial y}-x\frac{\partial}{\partial z},\quad \mathbf{e}_3=\frac{\partial}{\partial x}, \tag{2.1}$$

for which we have the Lie products

$$[\mathbf{e}_2,\mathbf{e}_3]=2\mathbf{e}_1,\quad [\mathbf{e}_3,\mathbf{e}_1]=0,\quad [\mathbf{e}_2,\mathbf{e}_1]=0, \tag{2.2}$$

with

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=1,\quad g(\mathbf{e}_3,\mathbf{e}_3)=-1. \tag{2.3}$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above, the following is true:*

$$\nabla=\begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2.4}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k=1,2,3\}=\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}.$$

We will use the notation

$$R_{abcd} = R(\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c, \mathbf{e}_d),$$

where the indices a, b, c and d take the values $1, 2$ and 3.

$$R_{1212} = -1, \quad R_{1313} = 1, \quad R_{2323} = -3. \tag{2.5}$$

3 Timelike Biharmonic Curves In The Lorentzian Heis^3

Let $\gamma : I \longrightarrow Heis^3$ be a timelike curve on the Lorentzian Heisenberg group Heis^3 parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group Heis^3 along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ is its torsion. With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T} \times \mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{3.2}$$

Lemma 3.1. *(see [18]) Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic timelike curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. γ is biharmonic if and only if*

$$\begin{aligned} \kappa &= \text{constant} \neq 0, \\ \tau &= \text{constant}, \\ N_1B_1 &= 0, \\ \kappa^2 - \tau^2 &= -1 + 4B_1^2. \end{aligned} \tag{3.3}$$

Theorem 3.2. *(see [18]) Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic timelike biharmonic curve on the Lorentzian Heisenberg group $Heis^3$ parametrized by arc length. If $N_1 = 0$, then*

$$\mathbf{T}(s) = \sinh\phi_0 \mathbf{e}_1 + \cosh\phi_0 \sinh\psi(s)\mathbf{e}_2 + \cosh\phi_0 \cosh\psi(s)\mathbf{e}_3, \tag{3.4}$$

where $\phi_0 \in \mathbb{R}$.

4 Bertrand Curves In The Lorentzian Heis^3

Definition 4.1. *A curve $\gamma : I \longrightarrow Heis^3$ with $\kappa \neq 0$ is called a Bertrand curve if there exist a curve $\widetilde{\gamma} : I \longrightarrow Heis^3$ such that the principal normal lines of γ and $\widetilde{\gamma}$ at $s \in I$ are equal. In this case $\widetilde{\gamma}$ is called a Bertrand mate of γ.*

Theorem 4.2. *Let $\gamma : I \longrightarrow Heis^3$ be a Bertrand curve. A Bertrand mate of γ is as follows:*

$$\widetilde{\gamma}(s) = \gamma(s) + \lambda \mathbf{N}(s), \quad \forall s \in I, \tag{4.1}$$

where λ is constant.

Proof. Using Definition 4.1 we have (4.1).

Theorem 4.3. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed timelike curve. If $\widetilde{\gamma}$ is a Bertrand mate of γ, then angle measurement of this curve between tangent vectors at corresponding points is constant.*

Proof. If we show $\left\langle \widetilde{\mathbf{T}}(s), \mathbf{T}(s) \right\rangle' = 0$, then the proof is complete.

$$\begin{aligned} \left\langle \widetilde{\mathbf{T}}(s), \mathbf{T}(s) \right\rangle' &= \left\langle \widetilde{\mathbf{T}}'(s), \mathbf{T}(s) \right\rangle + \left\langle \widetilde{\mathbf{T}}(s), \mathbf{T}'(s) \right\rangle \\ &= \left\langle \widetilde{\kappa}(s)\widetilde{v}(s)\widetilde{\mathbf{N}}(s), \mathbf{T}(s) \right\rangle + \left\langle \widetilde{\mathbf{T}}(s), \kappa(s)\mathbf{N}(s) \right\rangle \\ &= \widetilde{\kappa}(s)\widetilde{v}(s)\left\langle \widetilde{\mathbf{N}}(s), \mathbf{T}(s) \right\rangle + \kappa(s)\left\langle \widetilde{\mathbf{T}}(s), \mathbf{N}(s) \right\rangle. \end{aligned} \tag{4.2}$$

Since $\widetilde{\mathbf{N}}(s)$ is parallel to $\mathbf{N}(s)$ and $\mathbf{N}(s) \perp \mathbf{T}(s)$, then

$$\left\langle \widetilde{\mathbf{N}}(s), \mathbf{T}(s) \right\rangle = 0. \tag{4.3}$$

Since $\widetilde{\mathbf{T}}(s) \perp \widetilde{\mathbf{N}}(s)$ and $\widetilde{\mathbf{N}}(s)$ is parallel to $\mathbf{N}(s)$, then

$$\left\langle \widetilde{\mathbf{T}}(s), \mathbf{N}(s) \right\rangle = 0. \tag{4.4}$$

Substituting (4.3) and (4.4) in (4.2), we have

$$\left\langle \widetilde{\mathbf{T}}(s), \mathbf{T}(s) \right\rangle' = 0.$$

Hence, the proof is completed.

Theorem 4.4. *Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic timelike biharmonic curve If $\widetilde{\gamma}$ is a Bertrand mate of γ, then the parametric equations of $\widetilde{\gamma}$ is*

$$\begin{array}{rcl} x(s) & = & \frac{1}{\alpha}\cosh\phi_0 \sinh(\alpha s+\rho) + \lambda \sinh(\alpha s + C) + c_1, \\ y(s) & = & \frac{1}{\alpha}\cosh\phi_0 \cosh(\alpha s+\rho) + \lambda \cosh(\alpha s + C) + c_2, \\ z(s) & = & \sinh\phi_0 s - \frac{1}{2\alpha^2}\left[\cosh\phi_0\right]^2 \sinh 2(\alpha s+\rho) - \frac{[\cosh\phi_0]^2}{\alpha} s \\ & & -\frac{c_1}{\alpha}\cosh\phi_0 \cosh(\alpha s+\rho) + c_3 - \lambda\cosh(\alpha s + C)\sinh(\alpha s + C)), \end{array} \tag{4.5}$$

where $\alpha = \frac{|\kappa|}{\cosh\phi_0} - 2\sinh\phi_0$, $C = \rho - \arg\cosh\left[\frac{1}{\cosh\phi_0}(\kappa\sinh\phi_0 + \tau B_1)\right]$ and $\phi_0, c_1, c_2, c_3, \rho, \lambda \in \mathbb{R}$.

Proof. Let $\gamma(s) = (x(s), y(s), z(s))$ be a biharmonic curve parametrized by arc length. The covariant derivative of the vector field N given by (3.2) is

$$\nabla_{\mathbf{T}}\mathbf{N} = (T_2N_3 - T_3N_2)\,\mathbf{e}_1 + \left(N_2' + T_2N_3\right)\mathbf{e}_2 + \left(N_3' + T_1N_2\right)\mathbf{e}_3. \tag{4.6}$$

The first component of (4.6) is given by

$$< \nabla_{\mathbf{T}}\mathbf{N}, \mathbf{e}_1 >= T_2N_3 - T_3N_2. \tag{4.7}$$

On the other hand, using Frenet formulas (3.1), we have

$$< \nabla_{\mathbf{T}}\mathbf{N}, \mathbf{e}_1 >= \kappa T_1 + \tau B_1. \tag{4.8}$$

These, together with (4.7) and (4.8), give

$$\kappa T_1 + \tau B_1 = T_2N_3 - T_3N_2.$$

Since γ is parametrized by arc length and using $N_1 = 0$, we can write

$$N(s) = \cosh A(s)\mathbf{e}_2 + \sinh A(s)\mathbf{e}_3. \tag{4.9}$$

We shall take into account B_1 =constant, yields

$$\begin{aligned}\kappa \sinh \phi_0 + \tau B_1 &= \cosh \phi_0 \sinh(\alpha s + \rho) \sinh A(s) \\ &\quad - \cosh \phi_0 \cosh(\alpha s + \rho) \cosh A(s),\end{aligned}$$

or,

$$\kappa \sinh \phi_0 + \tau B_1 = -\cosh \phi_0 \cosh\left(\alpha s + \rho - A(s)\right).$$

Thus, we obtain

$$\cosh\left(\alpha s + \rho - A(s)\right) = -\frac{1}{\cosh \phi_0}\left(\kappa \sinh \phi_0 + \tau B_1\right) = \text{constant}. \tag{4.10}$$

From (4.10), we have

$$A(s) = \alpha s + C, \tag{4.11}$$

where

$$C = \rho - \arg\cosh\left[\frac{1}{\cosh \phi_0}\left(\kappa \sinh \phi_0 + \tau B_1\right)\right].$$

Substituting (4.11) in (4.9), we have

$$\mathbf{N}(s) = \cosh\left(\alpha s + C\right) e_2 + \sinh\left(\alpha s + C\right) e_3. \tag{4.12}$$

Using (2.1) and (4.1), we get

$$\begin{aligned}\widetilde{\gamma}(s) &= (\frac{1}{\alpha} \cosh \phi_0 \sinh(\alpha s + \rho) + c_1, \frac{1}{\alpha} \cosh \phi_0 \cosh(\alpha s + \rho) + c_2, \\ &\quad \sinh \phi_0 s - \frac{1}{2\alpha^2}\left[\cosh \phi_0\right]^2 \sinh 2(\alpha s + \rho) - \frac{\left[\cosh \phi_0\right]^2}{\alpha} s \\ &\quad -\frac{c_1}{\alpha} \cosh \phi_0 \cosh(\alpha s + \rho) + c_3) \\ &\quad +\lambda(\sinh\left(\alpha s + C\right), \cosh\left(\alpha s + C\right), -\cosh\left(\alpha s + C\right) \sinh\left(\alpha s + C\right)),\end{aligned}$$

or,

$$\begin{aligned}\widetilde{\gamma}(s) &= (\frac{1}{\alpha} \cosh \phi_0 \sinh(\alpha s + \rho) + \lambda \sinh\left(\alpha s + C\right) + c_1, \\ &\quad \frac{1}{\alpha} \cosh \phi_0 \cosh(\alpha s + \rho) + \lambda \cosh\left(\alpha s + C\right) + c_2, \\ &\quad \sinh \phi_0 s - \frac{1}{2\alpha^2}\left[\cosh \phi_0\right]^2 \sinh 2(\alpha s + \rho) - \frac{\left[\cosh \phi_0\right]^2}{\alpha} s \\ &\quad -\frac{c_1}{\alpha} \cosh \phi_0 \cosh(\alpha s + \rho) + c_3 - \lambda \cosh\left(\alpha s + C\right) \sinh\left(\alpha s + C\right)),\end{aligned}$$

where c_1, c_2, c_3 are constants of integration.

This implies (4.5). The proof is completed.

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Talat Körpınar
Fırat University, Department of Mathematics
23119, Elazığ, TURKEY
e-mail: talatkorpinar@gmail.com

Essin Turhan
Fırat University, Department of Mathematics
23119, Elazığ, TURKEY
e-mail: essin.turhan@gmail.com

Iqbal H. Jebril
Department of Mathematics, King Faisal University
Saudi Arabia
e-mail: iqbal501@yahoo.com