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Departamento de Matemáticas
Facultad de Ciencias
Universidad de Los Andes

A study of meromorphically starlike and convex functions

Imran Faisal and Ashfaq Khan

Abstract

In the present paper we introduce and study certain new subclasses of starlike and convex functions in the domain of meromorphic functions. Moreover we discuss coefficient inequalities, growth and distortion theorems, radii of starlikeness and convexity and convex linear combinations for the functions belonging to the newly introduced.

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1. Introduction and preliminaries

Let $\sum_r$ denote the class of functions f of the form,

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad z \in \mathcal{D}_r,\ a_n \geq o. \tag{1}$$

which are analytic in the punctured disk $\mathcal{D}_r = \{z : 0 < |z| < 1\}$.

Also let $\sum_r'$ denote the class of functions f of the form,

$$F(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^{n-\frac{n}{\alpha}} \quad \alpha \in N \setminus \{1\}, z \in \mathcal{D}_r, a_n \geq o \tag{2}$$

which are also analytic in the punctured disk $\mathcal{D}_r$(cf.,[18,19,20]). When α goes to infinity then $n - n/\alpha$ approaches to n where $\sum_r' = \sum_r$.

A function $F \in \sum_r'$ is called starlike of order $\beta(0 \leq \beta < 1)$ and is denoted by $M_r(\beta)$ if and only if

$$-Re(\frac{zF'(z)}{F(z)}) > \beta,\ z \in \mathcal{D}_r.$$

Similarly a function $F \in \sum_r'$ is called convex of order $\beta(0 \leq \beta < 1)$ and is denoted by $N_r(\beta)$ if and only if

$$-Re(1 + \frac{zf''(z)}{f'(z)}) > \beta, \quad z \in \mathcal{D}_r$$

Note that $F \in N_r(\beta) \Leftrightarrow -zF' \in M_r(\beta)$.

Many important properties and characteristics of various interesting subclasses of meromorphic functions such as starlike and convex functions were studied rather extensively by (amongothers) Uralgeddi[9], Aouf et.al([2][3]), Kulkarni et.al [7], Mogra([1],[8]) and Srivastava et.al ([5])(cf.,[6,10-17]. A summary of such papers is in the book by Srivastava and Owa [4].

2. Coefficient inequalities

Theorem 2.1. If $F \in \sum_r'$ and satisfy the following ineqality

$$\sum_1^\infty (k + n(\alpha - 1/\alpha) + |2\beta - k + n - n/\alpha|)\,|a_n|\,r^{n-n/\alpha+1} \le 2(1-\beta) \tag{3}$$

for some $\beta(0 \le \beta < 1)$ and $k(\beta < k \le 1)$, then $F \in M_r(\beta)$.

The result is attained for a function f given by

$$f(z) = \frac{1}{z} + \frac{2(1-\beta)}{(k + n(\alpha - 1/\alpha) + |2\beta - k + n - n/\alpha|)} z^{n-n/\alpha}.$$

Ozaki [6] has proved that a necessary and sufficient condition that $f \in \sum$ with $a_n \ge 0, (n = 1, 2, 3, ...)$ is meromorphic in D is that there should exist the relation

$$\sum_{n=1}^\infty n a_n z^{n+1} \le 1$$

between its coefficients.

Lemma 2.2. Let a function $F \in \sum_r'$ is also contain in the class $M_r(\beta)$ then,

$$\sum_1^\infty (k + n(\alpha - 1/\alpha) + |2\beta - k + n - n/\alpha|)\,|a_n|\,r^{n-n/\alpha+1} \le 2$$

Proof. Since $F \in \sum_r'$ implies

$$\sum_{n=1}^\infty n a_n z^{n+1} \le 1.$$

Therfore

$$\sum_{n=1}^\infty (k + n(\alpha - 1/\alpha) + |2\beta - k + n - n/\alpha|)\,|a_n|\,r^{n-n/\alpha+1}$$

$$= \sum_{n=1}^\infty (2n(\alpha - 1/\alpha) + 2\beta)\,|a_n|\,r^{n-n/\alpha+1}$$

$$= 2\sum_{n=1}^{\infty}(n(\alpha - 1/\alpha)\left|a_n\right| r^{n-n/\alpha+1} + \sum_{n=1}^{\infty}(2\beta)\left|a_n\right| r^{n-n/\alpha+1}$$

$$\leq 2(1) + (2\beta)(\frac{1}{n(1-1/\alpha)}) \leq 2, \quad \text{because} \quad n \to \infty$$

henca proved.

3. Growth and Distortion theorems

Theorem 3.1. If the functions F defined by (2) are in the class $M_r(\beta)$, then for $0 < |z| \leq 1$ we have

$$\frac{1}{r} - \frac{2(1-\beta)}{k+p+|2\beta-k+p|}r^p \leq |f(z)| \leq \frac{1}{r} + \frac{2(1-\beta)}{k+p+|2\beta-k+p|}r^p$$

where, $p = \alpha - 1/\alpha$ and equality holds for

$$f(z) = \frac{1}{z} + \frac{2(1-\beta)}{k+p+|2\beta-k+p|}z^p.$$

Proof. Since $F \in M_r(\beta)$, by using theorem 2.1, we have

$$\sum_{1}^{\infty}(k + n(\alpha - 1/\alpha) + |2\beta - k + n - n/\alpha|)\left|a_n\right| r^{n-n/\alpha+1} \leq 2(1-\beta)$$

thus, for $0 < |z| = r \leq 1$ we have

$$|F(z)| = \left|\frac{1}{z} + \sum_{n=1}^{\infty} a_n z^{n-n/\alpha}\right|,$$

$$\leq \frac{1}{|z|} + \sum_{n=1}^{\infty}\left|a_n\right| |z|^p,$$

$$\leq \frac{1}{r} + \frac{2(1-\beta)}{k+p+|2\beta-k+p|}r^p,$$

and,

$$|F(z)| = \left|\frac{1}{z} + \sum_{n=1}^{\infty} a_n z^{n-n/\alpha}\right|,$$

$$\geq \frac{1}{|z|} - \sum_{n=1}^{\infty}\left|a_n\right| |z|^p,$$

$$\geq \frac{1}{r} - \frac{2(1-\beta)}{k+p+|2\beta-k+p|}r^p,$$

This implies that

$$\frac{1}{r} - \frac{2(1-\beta)}{k+p+|2\beta-k+p|}r^p \leq |f(z)| \leq \frac{1}{r} + \frac{2(1-\beta)}{k+p+|2\beta-k+p|}r^p$$

Theorem 3.2. If the function F defined by (2) is in the class $M_r(\beta)$, then for $0<|z|\leq 1$ we have

$$\frac{1}{r^2}-\frac{2p(1-\beta)}{k+p+|2\beta-k+p|}r^{p-1}\leq\left|f'(z)\right|\leq\frac{1}{r^2}+\frac{2p(1-\beta)}{k+p+|2\beta-k+p|}r^{p-1}$$

where, $p=\alpha-1/\alpha$

Proof. Since $F\in M_r(\beta)$, by using theorem 2.1,we have

$$\sum_{1}^{\infty}(k+n(\alpha-1/\alpha)+|2\beta-k+n-n/\alpha|)\,|a_n|\,r^{n-n/\alpha+1}\leq 2(1-\beta)$$

Now by applying theorem 3.1,we have

$$\left|F'(z)\right|\leq\frac{1}{|z|^2}+\sum_{n=1}^{\infty}p\,|a_n|\,z^{p-1},$$

$$\leq\frac{1}{r^2}+\frac{2p(1-\beta)}{k+p+|2\beta-k+p|}r^{p-1},$$

Similarly,

$$\left|F'(z)\right|\geq\frac{1}{|z|^2}-\sum_{n=1}^{\infty}p\,|a_n|\,z^{p-1}\geq\frac{1}{r^2}-\frac{2p(1-\beta)}{k+p+|2\beta-k+p|}r^{p-1}$$

this implies,

$$\frac{1}{r^2}-\frac{2p(1-\beta)}{k+p+|2\beta-k+p|}r^{p-1}\leq\left|f'(z)\right|\leq\frac{1}{r^2}+\frac{2p(1-\beta)}{k+p+|2\beta-k+p|}r^{p-1}$$

as required.

Corollary 3.3. If $f(z)\in\sum_r'$ then, by lemma 2.2,

$$\frac{1}{r^2}-\frac{2p}{k+p+|2\beta-k+p|}r^{p-1}\leq\left|f'(z)\right|\leq\frac{1}{r^2}+\frac{2p}{k+p+|2\beta-k+p|}r^{p-1}$$

4. Radii of Starlikeness and Convexity

The radii of starlikeness and convexity for the functions belonging to the class $M_r(\beta)$, is given by the following theorem.

Theorem 4.1. If the function F defined by (2) is in the class $M_r(\beta)$, then f is starlike of order $\beta(0\leq\beta<1)$ in the unit disk $|z|<\gamma_1(\alpha,\beta,k,p)$, where $\gamma_1(\alpha,\beta,k,p)$, is the largest value for which

$$\gamma_1(\alpha,\beta,k,p)=inf(\frac{k+n-n/\alpha+|2\beta-k+n-n/\alpha|}{2(np+2-\beta})^{1/np+1},\quad p=1-1/\alpha$$

The result is sharp for functions f given by (3).

Proof. It suffices to show that

$$\left|zF'(z)/f(z)+1\right| \leq (1-\beta), \quad for\, |z| \leq r_1,,$$

since

$$F(z)=\frac{1}{z}+\sum_{n=1}^{\infty} a_n z^{np} \quad z\in\mathcal{D}_r,\ a_n\geq o, p=1-1/\alpha.$$

this implies,

$$\left|zF'(z)/F(z)+1\right| = \left|\frac{\sum_{n=1}^{\infty}(np+1)a_n z^{np+1}}{1+\sum_{n=1}^{\infty} a_n z^{np+1}}\right| \leq (1-\beta),$$

by using the theorem (1) we have,

$$\sum_{n=1}^{\infty}\left(\frac{2(np+1)(1-\beta)\,|z|^{np+1}}{k+n-n/\alpha+|2\beta-k+n-n/\alpha|}\right) \leq (1-\beta)-\sum_{n=1}^{\infty}\left(\frac{2(1-\beta)^2\,|z|^{np+1}}{k+n-n/\alpha+|2\beta-k+n-n/\alpha|}\right)$$

which implies,

$$\sum_{n=1}^{\infty}\left(\frac{2(np+2-\beta)\,|z|^{np+1}}{k+n-n/\alpha+|2\beta-k+n-n/\alpha|}\right) \leq 1$$

it follows that,

$$|z| \leq \left(\frac{k+n-n/\alpha+|2\beta-k+n-n/\alpha|}{2(np+2-\beta)}\right)^{1/np+1}, \quad n\geq 1$$

then,

$$r_1 = inf\left(\frac{k+n-n/\alpha+|2\beta-k+n-n/\alpha|}{2(np+2-\beta)}\right)^{1/np+1}, \quad n\geq 1$$

as required.

Theorem 4.2. If the function F defined by (2) is in the class $N_r(\beta)$, then f is starlike of order $\beta(0\leq\beta<1)$ in the unit disk $|z|<\gamma_2(\alpha,\beta,k,p)$, where $\gamma_2(\alpha,\beta,k,p)$, is the largest value for which

$$\gamma_2(\alpha,\beta,k,p) = inf\left(\frac{k+n-n/\alpha+|2\beta-k+n-n/\alpha|}{2np(np+1-\beta)}\right)^{1/np+1}, \quad p=1-1/\alpha$$

The result is sharp for functions f given by (3).

Proof. It suffices to show that,

$$\left|zF''(z)/F'(z)+2\right| \leq (1-\beta), \quad for \quad |z|\leq r_2,$$

$$\left|zF''(z)/F'(z)+2\right| = \left|\frac{\sum_1^{\infty}(np)^2 a_n z^{np+1}}{-1+\sum_1^{\infty} np a_n z^{np+2}}\right|$$

$$\leq \frac{\sum_1^\infty (np)^2(\frac{2(1-\beta)}{k+n-n/\alpha+|2\beta-k+n-n/\alpha|})\,|z|^{np+1}}{1-\sum_1^\infty (np)(\frac{2(1-\beta)}{k+n-n/\alpha+|2\beta-k+n-n/\alpha|})\,|z|^{np+1}}$$
$$\leq (1-\beta)$$

this implies that,

$$|z| \leq (\frac{k+n-n/\alpha+|2\beta-k+n-n/\alpha|}{2np(np+1-\beta)})^{1/np+1}, \quad n \geq 1$$

$$r_2 = inf(\frac{k+n-n/\alpha+|2\beta-k+n-n/\alpha|}{2np(np+1-\beta)})^{1/np+1}, \quad n \geq 1$$

5. Convex linear Combination

Our next result involve a linear combination for functions of the type given in (2).

Theorem 5.1. The class $M_r(\beta)$ is closed under convex linear combinations.

Proof. Suppose that the functions f_1 and f_2 defined by,

$$f_i(z) = \frac{1}{z} + \sum_1^\infty a_{n,i} z^{n-n/\alpha} \quad i = 1, 2$$

be in the class $M_r(\beta)$. Setting

$$f(z) = \mu f_1(z) + (1-\mu) f_2(z), \quad (0 \leq \mu \leq 1)$$

implies

$$f(z) = \frac{1}{z} + \sum_{n=1}^\infty (\mu a_{n,1} + (1-\mu) a_{n,2}) z^{n-n/\alpha}$$

in view of theorem 2.1, we have

$$\sum_{n=1}^\infty (k+n-n/\alpha+|2\beta-k+n-n/\alpha|)(\mu a_{n,1} + (1-\mu) a_{n,2})$$
$$= \mu \sum_{n=1}^\infty (k+n-n/\alpha+|2\beta-k+n-n/\alpha|)\,|a_{n,1}| +$$
$$(1-\mu) \sum_{n=1}^\infty (k+n-n/\alpha+|2\beta-k+n-n/\alpha|)\,|a_{n,2}|$$
$$\leq \mu(2(1-\beta)) + (1-\mu)(2(1-\beta))$$
$$\leq (1-\beta)(2\mu+1-2\mu)$$

$$\leq 2(1-\beta)$$

which show that

$$f(z) \in M_r(\beta)$$

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Imran Faisal
School of Mathematical Sciences,
Faculty of Science and Technology
Universiti Kebangsaan Malaysia,
Bangi 43600 Selangor D. Ehsan, Malaysia
e-mail: faisalmath@gmail.com

Ashfaq Khan
School of Business, Economics and Public Policy
University of New England,
Armidale, NSW 2351 Australia
e-mail: ashfaqmomi@yahoo.com, akhan27@une.edu.au