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Frenet Equations of Biharmonic Curves in terms of Exponential Maps in the Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

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Abstract

In this paper, we study biharmonic curves in the special three-dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$. Moreover, we construct matrix representation of biharmonic curves in terms of exponential maps in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Finally we obtain Frenet equations of biharmonic curves in terms of exponential maps in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$.

key words. Biharmonic curve, Para-Sasakian manifold, Exponential map.

AMS subject classifications. 53C41, 53A10.

1 Introduction

Differential geometry of curves is the branch of geometry that deals with smooth curves by methods of differential and integral calculus.

Starting in antiquity, many concrete curves have been thoroughly investigated using the synthetic approach. Differential geometry takes another path: curves are represented in a parametrized form, and their geometric properties and various quantities associated with them, such as the curvature and the arc length, are expressed via derivatives and integrals using vector calculus. One of the most important tools used to analyze a curve is the Frenet frame, a moving frame that provides a coordinate system at each point of the curve that is "best adapted" to the curve near that point.

The theory of curves is much simpler and narrower in scope than the theory of surfaces and its higher-dimensional generalizations, because a regular curve in a Euclidean space has no intrinsic geometry. Any regular curve may be parametrized by the arc length (the natural parametrization) and from the point of view of a bug on the curve that does not know anything about the ambient space, all curves would appear the same. Different space curves are only distinguished by the way in which they bend and twist. Quantitatively, this is measured by the differential-geometric invariants called the curvature and the torsion of a curve. The fundamental theorem of curves asserts that the knowledge of these invariants completely determines the curve.

In this paper, we study biharmonic curves in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Moreover, we construct matrix representation of biharmonic curves in terms of exponential maps in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Finally we obtain Frenet equations of biharmonic curves in terms of exponential maps in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$.

2 Preliminaries

An n-dimensional differentiable manifold M is said to admit an almost para-contact Riemannian structure (ϕ, ξ, η, g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric on M such that

$$\phi\xi = 0, \quad \eta(\xi) = 1, g(X, \xi) = \eta(X), \tag{2.1}$$

$$\phi^2(X) = X - \eta(X)\xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.3}$$

for any vector fields X, Y on M.

In addition, if (ϕ, ξ, η, g), satisfy the equations

$$d\eta = 0, \quad \nabla_X \xi = \phi X, \tag{2.4}$$

$$(\nabla_X \phi) Y = -g(X, Y)\xi - \eta(Y) X + 2\eta(X)\eta(Y)\xi, \qquad X, Y \in \chi(M), \tag{2.5}$$

then M is called a para-Sasakian manifold or, briefly a $P-$Sasakian manifold [2].

Definition 2.1. A para-Sasakian manifold M is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X, Y) Z) = 0,$$

for all vector fields X, Y, Z, W orthogonal to ξ. This notion was introduced by Takahashi [16], for a Sasakian manifold.

Definition 2.2. A para-Sasakian manifold M is said to be ϕ-symmetric if

$$\phi^2\left(\left(\nabla_W R\right)(X, Y)\, Z\right) = 0,$$

for all vector fields X, Y, Z, W on M.

Definition 2.3. A para-Sasakian manifold M is said to be ϕ-Ricci symmetric if the Ricci operator satisfies

$$\phi^2\left(\left(\nabla_X Q\right)(Y)\right) = 0,$$

for all vector fields X and Y on M and $S(X, Y) = g(QX, Y)$.

If X, Y are orthogonal to ξ, then the manifold is said to be locally ϕ-Ricci symmetric.

We consider the three-dimensional manifold

$$\mathbb{P} = \left\{\left(x^1, x^2, x^3\right) \in \mathbb{R}^3 : \left(x^1, x^2, x^3\right) \neq (0,0,0)\right\},$$

where $\left(x^1, x^2, x^3\right)$ are the standard coordinates in $\mathbb{R}^3$. We choose the vector fields

$$\mathbf{e}_1 = e^{x^1}\frac{\partial}{\partial x^2}, \quad \mathbf{e}_2 = e^{x^1}\left(\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^3}\right), \quad \mathbf{e}_3 = -\frac{\partial}{\partial x^1} \tag{2.6}$$

are linearly independent at each point of $\mathbb{P}$.

Let η be the 1-form defined by

$$\eta(Z) = g(Z, \mathbf{e}_3) \text{ for any } Z \in \chi(\mathbb{P}).$$

Let be the (1,1) tensor field defined by

$$\phi(\mathbf{e}_1) = \mathbf{e}_2,\ \phi(\mathbf{e}_2) = \mathbf{e}_1,\ \phi(\mathbf{e}_3) = 0.$$

Then using the linearity of and g we have

$$\eta(\mathbf{e}_3) = 1,$$

$$\phi^2(Z) = Z - \eta(Z)\mathbf{e}_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(\mathbb{P})$. Thus for $\mathbf{e}_3 = \xi$, (ϕ, ξ, η, g) defines an almost para-contact metric structure on $\mathbb{P}$, [2].

Let ∇ be the Levi-Civita connection with respect to g. Then, we have

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1, \quad [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_2.$$

Taking $\mathbf{e}_3 = \xi$ and using the Koszul's formula, we obtain

$$\begin{array}{lll} \nabla_{\mathbf{e}_1}\mathbf{e}_1 = -\mathbf{e}_3, & \nabla_{\mathbf{e}_1}\mathbf{e}_2 = 0, & \nabla_{\mathbf{e}_1}\mathbf{e}_3 = \mathbf{e}_1, \\ \nabla_{\mathbf{e}_2}\mathbf{e}_1 = 0, & \nabla_{\mathbf{e}_2}\mathbf{e}_2 = -\mathbf{e}_3, & \nabla_{\mathbf{e}_2}\mathbf{e}_3 = \mathbf{e}_2, \\ \nabla_{\mathbf{e}_3}\mathbf{e}_1 = 0, & \nabla_{\mathbf{e}_3}\mathbf{e}_2 = 0, & \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0. \end{array} \tag{2.7}$$

Moreover we put

$$R_{ijk} = R(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_k, \quad R_{ijkl} = R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l),$$

where the indices i, j, k and l take the values $1, 2$ and 3.

$$R_{122} = -\mathbf{e}_{1,}, \quad R_{133} = -\mathbf{e}_{1,}, \quad R_{233} = -\mathbf{e}_2,$$

and

$$R_{1212} = R_{1313} = R_{2323} = 1. \tag{2.8}$$

3 New Approach for Biharmonic Curves in $\mathbb{P}$

A map

$$\exp : R \times \mathbb{P}_3^3 \to GL(3, \mathbb{R}) \subset \mathbb{P}_3^3, \quad (t, \mathcal{A}) \to \exp(t, \mathcal{A}) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathcal{A}^k$$

is called exponential map in Para-Sasakian Manifold $\mathbb{P}$.

Definition 3.1. $\langle \mathcal{A}, \mathcal{B} \rangle_{\mathbb{P}} = \mathfrak{trace}\left(\mathcal{A}\mathcal{B}^T\right)$ *is called an inner product for* $\mathcal{A}, \mathcal{B} \in \mathbb{P}_3^3$.

Firstly, let us calculate the arbitrary parameter t according to the arclength parameter s. It is well known that

$$s = \int_0^t \left\| \gamma'(t) \right\|_{\mathbb{P}} dt, \tag{3.1}$$

where

$$\gamma'(t) = \mathcal{A}\gamma. \tag{3.2}$$

The norm of Equation (3.1), we obtain

$$\|\mathcal{A}\gamma\|_{\mathbb{P}} = \sqrt{-\mathfrak{trace}\,(\mathcal{A}^2)}.$$

Substituting above equation in (3.1), we have

$$s = \sqrt{-\mathfrak{trace}\,(\mathcal{A}^2)}t$$

Lemma 3.2. *Let* $\mathcal{A}$ *be a be an anti-symmetric matrix and* $n \in \mathbb{N}$. *Then,*

i) *If* n *is odd,* $\mathcal{A}^n$ *is an anti-symmetric matrix.*

ii) *If* n *is even,* $\mathcal{A}^n$ *is a symmetric matrix.*

iii) *The trace of an anti-symmetric matrix is zero.*

The first, second and third derivatives of γ are given as follows:

$$\gamma'(s) = \frac{\mathcal{A}\gamma}{\sqrt{-\mathfrak{trace}\,(\mathcal{A}^2)}},\ \gamma''(s) = \frac{\mathcal{A}^2\gamma}{\left(\sqrt{-\mathfrak{trace}\,(\mathcal{A}^2)}\right)^2},\ \gamma'''(s) = \frac{\mathcal{A}^3\gamma}{\left(\sqrt{-\mathfrak{trace}\,(\mathcal{A}^2)}\right)^3}. \tag{3.3}$$

4 New Frenet Frame of Biharmonic Curves in terms of Usual Frenet Vectors in the Special Three-Dimensional $\phi-$Ricci Symmetric Para-Sasakian Manifold $\mathbb{P}$

Using above sections we obtain following results.

Theorem 4.1. *(see [12]) Let $\gamma : I \longrightarrow \mathbb{P}$ be a unit speed non-geodesic biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Then,*

$$\begin{aligned}\mathcal{A}\gamma &= \sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}(-\cos\varphi, \sin\varphi e^{-s\cos\varphi + C_1}\left(\sin\left[\Bbbk s + C\right] + \cos\left[\Bbbk s + C\right]\right), \\ &\quad \sin\varphi e^{-s\cos\varphi + C_1}\sin\left[\Bbbk s + C\right]),\end{aligned}$$

$$\begin{aligned}\mathcal{A}^2\gamma = \frac{\left(\sqrt{\mathfrak{trace}(\mathcal{A}^4)}\right)}{\kappa}&(-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2, \\ &e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(\Bbbk\sin\varphi\sin\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\cos\left[\Bbbk s + C\right]\right) \\ &+ e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(-\Bbbk\sin\varphi\cos\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\sin\left[\Bbbk s + C\right]\right), \\ &- e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(-\Bbbk\sin\varphi\cos\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\sin\left[\Bbbk s + C\right]\right)),\end{aligned} \tag{4.1}$$

$$\begin{aligned}\mathcal{A}^3\gamma &= \frac{1}{\kappa}\left[\frac{\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^3} - \frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^2}\right]^{\frac{1}{2}}(-\sin\varphi e^{-s\cos\varphi + C_1}(\sin\left[\Bbbk s + C\right] \\ &\quad + \cos\left[\Bbbk s + C\right])e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}.(-\Bbbk\sin\varphi\cos\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\sin\left[\Bbbk s + C\right]) \\ &\quad - \sin\varphi e^{-s\cos\varphi + C_1}\sin\left[\Bbbk s + C\right]e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}(\Bbbk\sin\varphi\sin\left[\Bbbk s + C\right] + \\ &\quad + \cos\varphi\sin\varphi\cos\left[\Bbbk s + C\right]) \\ &\quad + (-\Bbbk\sin\varphi\cos\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\sin\left[\Bbbk s + C\right])), \\ &\quad (-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2)\sin\varphi e^{-s\cos\varphi + C_1}\sin\left[\Bbbk s + C\right] \\ &\quad - \cos\varphi e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}\left(-\Bbbk\sin\varphi\cos\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\sin\left[\Bbbk s + C\right]\right), \\ &\quad - \cos\varphi e^{-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2}((\Bbbk\sin\varphi\sin\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\cos\left[\Bbbk s + C\right]) \\ &\quad + (-\Bbbk\sin\varphi\cos\left[\Bbbk s + C\right] + \cos\varphi\sin\varphi\sin\left[\Bbbk s + C\right])) \\ &\quad - \sin\varphi e^{-s\cos\varphi + C_1}\left(\sin\left[\Bbbk s + C\right] + \cos\left[\Bbbk s + C\right]\right)(-\frac{\sin^2\varphi}{2}s^2 + \overline{C}_1 s + \overline{C}_2) \\ &\quad - \frac{\mathfrak{trace}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)}(-\cos\varphi, \sin\varphi e^{x^1}\left(\sin\left[\Bbbk s + C\right] + \cos\left[\Bbbk s + C\right]\right), \sin\varphi e^{x^1}\sin\left[\Bbbk s + C\right]),\end{aligned}$$

where $C, \overline{C}_1, \overline{C}_2$ *are constants of integration and* $\Bbbk = \frac{\sqrt{\kappa^2 - \sin^2\varphi}}{\sin\varphi}$.

In the light of Theorem 4.1, we also give the following theorems :

Theorem 4.2. *Let $\gamma : I \longrightarrow \mathbb{P}$ be a unit speed non-geodesic biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Then the new Frenet equations of this curve are*

$$\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T} &= -\frac{\mathcal{A}^2\gamma}{\mathfrak{trace}(\mathcal{A}^2)}, \\
\nabla_{\mathbf{T}}\mathbf{N} &= \frac{\mathcal{A}^3\gamma}{\sqrt{-\mathfrak{trace}(\mathcal{A}^2)}\sqrt{\mathfrak{trace}(\mathcal{A}^4)}}, \\
\nabla_{\mathbf{T}}\mathbf{B} &= [\frac{\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}(\mathcal{A}^2)}\right)^6} - \frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathfrak{trace}(\mathcal{A}^2)}\right)^5}]^{-\frac{1}{2}}[\frac{\mathcal{A}^4\gamma}{\left(\sqrt{-\mathfrak{trace}(\mathcal{A}^2)}\right)^4} \\
&\quad + \frac{\mathfrak{trace}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\mathfrak{trace}(\mathcal{A}^2)}\right)^6}\mathcal{A}^2\gamma].
\end{aligned} \tag{4.2}$$

Proof. We assume that γ is a unit speed non-geodesic biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$.

From the proof of above Theorem we obtain

$$\mathbf{T} = \frac{\mathcal{A}\gamma}{\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}} \tag{4.3}$$

So, by differentiating of the formula (13), we get

$$\nabla_{\mathbf{T}}\mathbf{T} = -\frac{\mathcal{A}^2\gamma}{\mathfrak{trace}\left(\mathcal{A}^2\right)}.$$

Also, we have the principal normal of the curve

$$\mathbf{N} = \frac{\mathcal{A}^2\gamma}{\sqrt{\mathfrak{trace}\left(\mathcal{A}^4\right)}}. \tag{4.4}$$

Differentiating of the formula (4.4), we get

$$\nabla_{\mathbf{T}}\mathbf{N} = \frac{\mathcal{A}^2\gamma'}{\sqrt{\mathfrak{trace}\left(\mathcal{A}^4\right)}}.$$

Using (4.3) in above equation, we have

$$\nabla_{\mathbf{T}}\mathbf{N} = \frac{\mathcal{A}^3\gamma}{\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\sqrt{\mathfrak{trace}\left(\mathcal{A}^4\right)}}.$$

Finally, the same above method we will find $\nabla_{\mathbf{T}}\mathbf{B}$. We have the binormal of the curve

$$\mathbf{B} = [\frac{\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^6} - \frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}]^{-\frac{1}{2}}[\frac{\mathcal{A}^3\gamma}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^3} + \frac{\mathfrak{trace}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}\mathcal{A}\gamma]. \tag{4.5}$$

Also, by differentiating of the formula (4.5), we get

$$\nabla_{\mathbf{T}}\mathbf{B} =[\frac{\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^6}-\frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}]^{-\frac{1}{2}}[\frac{\mathcal{A}^3\gamma'}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^3}+\frac{\mathfrak{trace}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}\mathcal{A}\gamma'].$$

Since, we have

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{B} &= [\frac{\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^6}-\frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}]^{-\frac{1}{2}}[\frac{\mathcal{A}^3}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^3}\frac{\mathcal{A}\gamma}{\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}}\\ &\quad+\frac{\mathfrak{trace}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}\mathcal{A}\frac{\mathcal{A}\gamma}{\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}}].\end{aligned}$$

So we immediately arrive at

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{B} &= [\frac{\mathfrak{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^6}-\frac{\left(\mathfrak{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^5}]^{-\frac{1}{2}}[\frac{\mathcal{A}^4\gamma}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^4}\\ &\quad+\frac{\mathfrak{trace}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^6}\mathcal{A}^2\gamma].\end{aligned}$$

This completes the proof of the theorem.

In the light of Theorem 4.1 and Theorem 4.2, we express the following corollary without proof:

Corollary 4.3. *Let $\gamma : I \longrightarrow \mathbb{P}$ be a unit speed non-geodesic biharmonic curve in the special three-dimensional $\phi-$Ricci symmetric para-Sasakian manifold $\mathbb{P}$. Then*

$$\begin{aligned}\mathcal{A}^4\gamma &= -[\frac{\left(\sqrt{-\mathfrak{trace}(\mathcal{A}^2)}\right)^4\tau}{\sqrt{\mathfrak{trace}(\mathcal{A}^4)}}[\frac{\mathfrak{trace}(\mathcal{A}^6)}{\left(\sqrt{-\mathfrak{trace}(\mathcal{A}^2)}\right)^6}-\frac{\left(\mathfrak{trace}(\mathcal{A}^4)\right)^2}{\left(\sqrt{-\mathfrak{trace}(\mathcal{A}^2)}\right)^5}]^{\frac{1}{2}}\\ &\quad-\mathfrak{trace}\left(\mathcal{A}^4\right)\left(\sqrt{-\mathfrak{trace}\left(\mathcal{A}^2\right)}\right)^{-2}]\frac{\left(\sqrt{\mathfrak{trace}(\mathcal{A}^4)}\right)}{\kappa}(-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1s+\overline{C}_2,\\ &\quad e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1s+\overline{C}_2}\left(\Bbbk\sin\varphi\sin\left[\Bbbk s+C\right]+\cos\varphi\sin\varphi\cos\left[\Bbbk s+C\right]\right)\\ &\quad+e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1s+\overline{C}_2}\left(-\Bbbk\sin\varphi\cos\left[\Bbbk s+C\right]+\cos\varphi\sin\varphi\sin\left[\Bbbk s+C\right]\right),\\ &\quad-e^{-\frac{\sin^2\varphi}{2}s^2+\overline{C}_1s+\overline{C}_2}\left(-\Bbbk\sin\varphi\cos\left[\Bbbk s+C\right]+\cos\varphi\sin\varphi\sin\left[\Bbbk s+C\right]\right)),\end{aligned} \tag{4.6}$$

where C, $\overline{C}_1, \overline{C}_2$ are constants of integration and $\Bbbk = \frac{\sqrt{\kappa^2-\sin^2\varphi}}{\sin\varphi}$.

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